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MONTE CARLO COMPUTATION OF THE STATISTICS OF THE MIDCOURSE VELOCITY CORRECTIONS FOR A LUNAR MISSION

by Gerald L. Smith and Burnett L. Gadeberg

Ames Research Center

Moffett Field, Calif.

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SUMMARY

A Monte Carlo method is described for obtaining the statistics of the total velocity correction employed in the multicorrection midcourse guidance of a space vehicle. The problem is analyzed to show the statistical correlation which exists between successive corrections and to develop equations necessary for implementing a Monte Carlo computer program. Covariance matrices of the individual corrections, computed by means of prior simulation of the problem, are required as inputs to the program.

Results are given for the application of the technique to the midcourse guidance phase of a circumlunar flight having five velocity corrections. Analysis of the results indicates that commonly used estimates of fuel requirements from calculated rms velocity corrections may result in an excessive fuel load.

INTRODUCTION

The problem of determining the amount of fuel required for midcourse guidance of a lunar or interplanetary vehicle can be a critical one. Sufficient fuel must be provided to ensure the probability of mission success; however, too much fuel could mean that the payload is diminished and the mission less profitable. Thus, methods for accurately computing the fuel requirement are of considerable interest.

In general, this is a statistical problem since the factors which affect fuel usage on a particular flight are random variables - namely, the injection errors, navigation errors, and velocity correction implementation errors. Thus, the exact amount of fuel needed for the mission cannot be computed beforehand and fuel tankage must be based on statistical averages.

A common procedure which has been used for establishing fuel requirements is to (1) obtain rms values for the individual velocity corrections by a computer simulation similar to that described in references 1 and 2, (2) add these to obtain an rms figure for the total ΔV, and (3) assume three times this total as the amount of velocity correction to plan on for an adequate probability of success. Often the success probability figure quoted for this method is 99.74 percent, which is the three-sigma figure for a gaussian distribution. There are two things wrong with this procedure. First, when there is correlation between successive velocity corrections, the rms value of the total is less than the sum of the rms values of the individual corrections. Second, since the total ΔV has a nonzero mean and is nongaussian, the success probability is actually greater than 99.74 percent. Thus, this procedure will result in estimating a greater amount of fuel than is necessary for a specified mission success probability.

To obtain a better measure of fuel requirements, a more accurate method of determining the statistics of the velocity corrections is necessary. In this paper, a practical Monte Carlo technique is described which can give the desired statistical information. Also, by examination of some results obtained using this method, an approximate rule similar to the three-sigma gaussian rule is developed for interpreting rms figures in terms of probability of success.

NOTATION

B_i	guidance law matrix for ith velocity correction
$E[\ ]$	expected value of $[\ ]$
H	a portion (submatrix) of M
K	weighting matrix in estimation equations
M	matrix of partial derivatives in estimation equations
n	noise, or error, in observations
$p(v_1,\ldots,v_n)$	joint probability density function of the random variables $v_i,\ldots,v_n$
Q	matrix of eigenvectors
$(\ )^T$	transpose of a matrix
t	time
ΔV	the sum of the magnitudes of the v_i
ΔV_{rms}	rms value of ΔV
v_{ce_i}	error in implementing v_i
v_i	ith velocity correction

v_{m_i}	measurement of v_i
v_{me_i}	error in measurement of v_i
v_{rms}	rms value of v
x	state vector (vehicle position and velocity)
x^*	augmented state vector
$\tilde{x}$	error in estimate of x, $\hat{x} - x$
$\hat{x}_i$	estimate of x based on all observations preceding the ith velocity correction
$\hat{x}'_i$	estimate of x just after the ith velocity correction
$\Delta\hat{x}_i$	change in $\hat{x}$ due to observations in the interval between the (i - 1) and ith velocity correction
y	observation vector
σ	standard deviation
Φ_i	state transition matrix between the (i - 1) and ith velocity correction

ANALYSIS

Statement of the Problem

The analysis presented herein can apply to the midcourse phase of any type of space flight in which impulsive velocity corrections are employed. However, for the purpose of illustrating the method, we assume here a circumlunar mission and a self-contained on-board navigation and guidance system of the type described in references 1, 2, and 3. The trajectory of the example case and the schedule of observations and velocity corrections employed are shown in figure 1.

The five velocity correction vectors in this schedule may be defined as $v_1, v_2, \ldots, v_5$. Then define

$$\Delta V = |v_1| + |v_2| + \ldots + |v_5| \tag{1}$$

as the total midcourse correction. The amount of fuel used in the five corrections is proportional to ΔV if the mass of the vehicle is constant.[1] For a statistical description of the guidance system performance, the statistics of ΔV are required. For instance, to determine the amount of fuel tankage to be provided for midcourse guidance fuel, one might want to know the probability that ΔV will not exceed a given value, that is, $\Pr[\Delta V \leq R]$.

In all that follows it is useful to make the reasonable assumption that the individual midcourse velocity corrections have multivariate gaussian distributions since this makes it possible to describe the distributions completely by covariance matrices. Furthermore, even if the v_i are not gaussian, only the covariance matrices are required if merely second-order statistics of ΔV are to be obtained. Thus, the results obtained herein are universally correct for gaussian v_i and correct in regard to second-order statistics for arbitrary v_i distributions.

Description of ΔV Statistics by Means of a Joint Density Function

Computing ΔV statistics by any method requires a knowledge, either explicit or implicit, of the joint probability density function, $p(v_1,\ldots,v_5)$. If the v_i are gaussian with zero mean, then this density function is completely described by the covariance matrix

$$Evv^T = \begin{bmatrix} Ev_1v_1^T & Ev_1v_2^T & \cdot & \cdot & \cdot & Ev_1v_5^T \\ Ev_2v_1^T & Ev_2v_2^T & \cdot & \cdot & \cdot & \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ Ev_5v_1^T & & & & & Ev_5v_5^T \end{bmatrix} \tag{2}$$

The (3×3) submatrices along the diagonal, $Ev_iv_i^T$, are the covariance matrices of the individual velocity corrections. The off-diagonal (3×3) submatrices depend upon the correlation between different velocity corrections. Reference 2 gives the equations for the numerical computation of the $Ev_iv_i^T$, but some additional analysis, which will be given later, is required to indicate how the other covariance matrices may be computed.

Before proceeding with the analysis we will find it useful to discuss the physical reasons for the existence of correlation between the individual velocity corrections. Such correlation will exist whenever the knowledge of the vehicle state following a correction is sufficient to compute a nonzero velocity correction at a later time. In the present problem this condition may arise in any one of three ways. First, each indicated (or commanded) correction is imperfectly executed. However, since the correction is

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[1]For a nonconstant mass vehicle, total fuel is proportional to a weighted sum of the $|v_i|$. This is a somewhat more complicated problem but one which can be solved by application of the techniques herein described.

monitored, the error is assumed known[2] and thus contributes to the corrections made at succeeding times. Secondly, the guidance law employed by the guidance system is generally designed to null some component of miss at a future point. If at any time during the flight this aim point is changed, information is already present within the system, without further observations of the trajectory, to make a correction based on the new aim point. For instance, in the circumlunar mission guidance system described in reference 2, the initial aim point is a prescribed perilune, and velocity corrections are computed to null the estimated position deviation at this point, without regard to the velocity deviation which will exist upon arrival. When perilune is reached, the aim point is then changed to a prescribed virtual perigee. The uncorrected velocity deviation at perilune obviously would produce a miss at perigee which could be corrected, within the knowledge available to the system. This correction then appears as part of the first correction after perilune passage and is, of course, correlated with all the corrections made before selection of the new aim criterion.

The third situation in which correlation between corrections may arise is when all of the indicated correction is not applied. This is a matter of guidance logic which does not apply in the present study because the assumed guidance law is a full-correction scheme identical to that described in reference 2.

The problem is now to develop equations for the covariance matrices $Ev_i v_j^T$ which describe the correlation between individual velocity corrections. It is of interest to note that in another paper on the statistics of midcourse velocity corrections (ref. 4), the true character of this correlation is not recognized. The error in reference 4 will be apparent later in our analysis, specifically in the development given in the appendix where the nature of the error will be discussed.

To begin the analysis, we note that the first velocity correction can be described by the expression[3]

$$v_1 = B_1\hat{x}_1 + v_{ce_1} \tag{3}$$

where B_1 is the guidance law matrix at time t, $\hat{x}_1$ is the estimated state vector at time t, and v_{ce_1} is the error in executing the correction. After the correction, the estimate $\hat{x}$ becomes

$$\left.\begin{aligned} \hat{x}_1' &= \hat{x}_1 + v_{m_1} \\ &= \hat{x}_1 + v_1 - v_{me_1} \\ &= (I + B_1)\hat{x}_1 + v_{ce_1} - v_{me_1} \end{aligned}\right\} \tag{4}$$

[2]Errors in measuring or monitoring the corrections are assumed negligible in this paper.

[3]In equation (3) and those which follow, the velocity vectors are regarded formally as six-component state vectors with zero values for the first three (position) components. The B matrix is then formally a (6×6) with zeros in the first three rows.

where v_{m_1} is the measurement of the correction, and v_{me_1} is the error in measuring the correction. In the present study it is assumed that $v_{me} \ll v_{ce}$, so that from this point on for practical simplicity v_{me_1} is dropped, and equation (4) is rewritten

$$\hat{x}_1' = (I + B_1)\hat{x}_1 + v_{ce_1} \tag{5}$$

Note that $\hat{x}_1$ and v_{ce_1} are (by assumption) uncorrelated random variables, the covariance matrices of which are computed in the machine programs used to obtain the results reported in references 1 and 2.

The equation for v_2 is of the same form as equation (3):

$$v_2 = B_2\hat{x}_2 + v_{ce_2} \tag{6}$$

and, just as for the first velocity correction, $\hat{x}_2$ and v_{ce_2} are assumed uncorrelated. However, $\hat{x}_2$ in general is correlated with v_1. That is, if $\hat{x}_1'$ from equation (5) were updated to the time of the second correction by means of the 6×6 transition matrix $\Phi(t_2,t_1)$, it could be used to compute a velocity correction $B_2\Phi(t_2,t_1)\hat{x}_1'$, which is obviously correlated with v_1. In fact, this is the only part of v_2 which is correlated with v_1, as is shown in the appendix. It is, therefore, convenient for the purposes of our analysis to rewrite equation (6) as

$$v_2 = B_2[\hat{x}_2 - \Phi(t_2,t_1)\hat{x}_1'] + B_2\Phi(t_2,t_1)\hat{x}_1' + v_{ce_2} \tag{7}$$

It is seen that the bracketed term in (7) can be described as the change in the state estimate due to the sequence of observations between the first and second velocity corrections. (Note that if there were no observations, $\hat{x}_2$ would be equal to the updated estimate $\Phi(t_2,t_1)\hat{x}_1'$ and the bracketed term would then be zero.) This term is thus appropriately defined as

$$\Delta\hat{x}_2 = [\hat{x}_2 - \Phi_2\hat{x}_1'] \tag{8}$$

where the shorthand notation $\Phi(t_2,t_1) = \Phi_2$ has been employed. It may be further noted that since the estimate is assumed to be zero at the beginning of the midcourse guidance problem, say at injection, $\hat{x}_1$ can be defined likewise as a change in the state estimate,

$$\Delta\hat{x}_1 = \hat{x}_1 \tag{9}$$

If definitions (8) and (9) are used, and equation (5) is substituted for $\hat{x}_1'$ in the second term on the right of equation (7), the expressions for the first two velocity corrections become:

$$v_1 = B_1\Delta\hat{x}_1 + v_{ce_1} \tag{10}$$

$$\begin{aligned} v_2 = {} & B_2\Phi_2(I + B_1)\Delta\hat{x}_1 + B_2\Phi_2 v_{ce_1} \\ & + B_2\Delta\hat{x}_2 + v_{ce_2} \end{aligned} \tag{11}$$

Continuing in like manner for the third correction, one obtains

$$v_3 = B_3\Phi_3(I + B_2)\Phi_2(I + B_1)\Delta\hat{x}_1 + B_3\Phi_3(I + B_2)\Phi_2 v_{ce_1}$$
$$+ B_3\Phi_3(I + B_2)\Delta\hat{x}_2 + B_3\Phi_3 v_{ce_2}$$
$$+ B_3\Delta\hat{x}_3 + v_{ce_3} \tag{12}$$

The pattern is apparent. Each velocity correction can be written in terms of the random vectors $\Delta\hat{x}_i$ and v_{ce_i}. The random vectors on the right side of equations (10), (11), and (12) are all statistically uncorrelated with each other, a property which is proved in the appendix.

With the expressions for the velocity corrections $v_1,\ldots,v_5$ written in terms of the ten uncorrelated random vectors $\Delta\hat{x}_1,\ldots,\Delta\hat{x}_5$, $v_{ce_1},\ldots,v_{ce_5}$, equations for the covariance matrices $Ev_iv_j^T$ in (2) can be written quite easily in terms of the covariance matrices $E\Delta\hat{x}_i\Delta\hat{x}_i^T$ and $Ev_{ce_i}v_{ce_i}^T$, $i = 1,\ldots 5$. For instance,

$$Ev_1v_1^T = B_1[E\Delta\hat{x}_1\Delta\hat{x}_1^T]B_1^T + [Ev_{ce_1}v_{ce_1}^T] \tag{13}$$

$$Ev_1v_2^T = B_1[E\Delta\hat{x}_1\Delta\hat{x}_1^T](I + B_1)^T\Phi_2^TB_2^T$$
$$+ [Ev_{ce_1}v_{ce_1}^T]\Phi_2^TB_2^T \tag{14}$$

Numerical values for these covariance matrices can be computed for any particular assumed situation as an adjunct to the machine program used to simulate the midcourse guidance problem. Thereby, the covariance matrix Evv^T of expression (2) can be constructed.

Numerical Integration to Obtain ΔV Statistics

Having the joint density function, determined as described above, one may proceed to the computation of any desired statistic of ΔV by integration over the 15-dimensional space of the v_i. For instance, the mean value of ΔV is obtained from the evaluation of the multiple integral

$$E\Delta V = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Delta V p(v_1, \ldots, v_5)\, dv_{11}, \ldots, dv_{55}$$

Of course, in general such an integration would have to be done numerically and the results would not likely be very satisfactory, either in regard to accuracy or the machine time required for the job. This difficulty provides the motivation for employing a Monte Carlo approach as described in the next section.

The Monte Carlo Method

The Monte Carlo method is a scheme that generates ΔV samples which are consistent with the precomputed statistics of the random variables v_i. From a large collection of such ΔV samples, any statistical parameter desired may be calculated by a straightforward statistical analysis.

Actual simulation of many complete flights for generating such samples would be unthinkable because of the huge amount of computing time required. Fortunately, this is not necessary. In a single simulation run the statistics (covariance matrices) of all the uncorrelated random variables which determine ΔV can be computed. All that is needed then is to randomly pick values for these variables consistent with the computed statistics and compute a ΔV using the functional relationship (1) between these variables and ΔV. This is a relatively simple computational procedure and may be repeated many thousands of times without using excessive machine time.

Equations (10), (11), (12), etc., could be used in the machine program. However, it is a bit simpler to employ more basic expressions derived from equations (3), (4), and (8). For each velocity correction the following set of equations applies:

$$\hat{x}_i = \Delta\hat{x}_i + \Phi_i\hat{x}'_{i-1} \tag{15}$$

$$v_i = B_i\hat{x}_i + v_{ce_i} \tag{16}$$

$$\hat{x}'_i = \hat{x}_i + v_i \tag{17}$$

The computation of a sample ΔV begins with the first correction $(i = 1)$, where $\Delta\hat{x}_{i-1} = \hat{x}_0 = 0$ by assumption. A random $\Delta\hat{x}_1$ and v_{ce_1} are generated and used in equations (15) and (16). The computed $\hat{x}_1$ from equation (17) is then used, together with new random variables $\Delta\hat{x}_2$ and v_{ce_2}, to compute v_2 and $\hat{x}_2$, and the process repeated through all the corrections. The total correction ΔV is then formed from

$$\Delta V = \sum \left| v_i \right| \tag{18}$$

and stored. Repeating this sequence many times gives the required ΔV collection.

As stated above, the Monte Carlo computations require the selection of random vectors consistent with the second-order statistics of the random variables which comprise the velocity corrections. The procedure for doing this is as follows. We start with a random number generator (available as a digital computer subroutine), which we may incorporate into the computer program. This subroutine generates a sequence of numbers having a gaussian distribution with zero mean and unit standard deviation. The numbers are all independent, that is, uncorrelated with one another. A set of six of these numbers may be regarded as a six-component random vector. Since the six components are uncorrelated with each other, such a vector has, by definition, a diagonal covariance matrix. Also this matrix has unit elements on the diagonal because the individual random numbers have unit variance. Thus, if u is the random vector, its covariance matrix U is

$$Euu^T = U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

Now it is apparent that by scaling the random numbers which comprise the components of u by, say, multiplying the first by a constant σ_1, the second by σ_2, etc., there is obtained a random vector w with covariance matrix

$$Eww^T = W = \begin{bmatrix} \sigma_1^2 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \sigma_2^2 & & & & \\ \cdot & & \cdot & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ 0 & & & & & \sigma_6^2 \end{bmatrix} \tag{20}$$

Furthermore, if we apply a linear transformation to the scaled vector,

$$w' = Qw \tag{21}$$

we obtain a new (related) random vector whose covariance matrix is not diagonal:

$$\left.\begin{aligned} Ew'w'^T &= QEww^TQ^T \\ &= QWQ^T = W' \end{aligned}\right\} \tag{22}$$

Thus, starting with independent gaussian random numbers, random vectors can be constructed having any desired statistical properties (as expressed by a covariance matrix).

In order to use this procedure it is necessary to diagonalize the covariance matrices of each of the random variables in the problem to obtain the σ scale factors and the Q transformation matrices to be used in generating the $\Delta\hat{x}_i$ and v_{ce_i} from the random numbers. Thus, a diagonalizing routine must be part of the Monte Carlo machine program.

In summary, the Monte Carlo program consists of the following sections:

(1) Random number generator
(2) Diagonalizing routine
(3) Equations (15), (16), and (17)
(4) Statistical analysis computations

Inputs to the program are the covariance matrices of the $\Delta\hat{x}_i$ and v_{ce_i} random variables, and the Φ_i transition matrices and B_i guidance law matrices. These are computed beforehand in a run of the complete simulated guidance problem.

RESULTS AND DISCUSSION

A FORTRAN 7094 computer program was written to test the principles outlined in the Analysis section and to obtain some numerical results for use in studying the general characteristics of the statistics of midcourse velocity corrections. The program will not be described in detail here, except to state that the statistics computed were the distribution function, the density function, the mean, and the rms value of each of the velocity corrections and of the total ΔV.

The circumlunar mission employed as an example has been described previously. The assumed nominal circumstances pertinent to the present problem are:

Correction mechanization errors (rms values):
1 percent in magnitude of the correction
1° in direction
0.1 m/sec in cut-off

Injection errors (rms values):
1 km and 1 m/sec in each of the three directions in a geocentric coordinate system

Test results were obtained for this nominal situation and three variations thereon:

(1) No velocity correction mechanization errors
(2) Twice-nominal correction mechanization errors
(3) Five-times-nominal injection errors

These cases were expected to give a fair idea of the effect on ΔV statistics of correlation between successive velocity corrections.

Figures 2 to 5 show the sample distribution and density functions for each of the velocity corrections and for the total ΔV, for each of the four conditions described. Each run has a sample size of 5000. Table I gives the 99-percent probability points (taken from the sample distribution curves), the means, and rms values. The theoretical rms values (obtained from the covariance matrices) are also given for the individual corrections.

A general comment which may be made regarding the density functions is that they are noticeably nongaussian, so that gaussian approximations are apt to be considerably in error, especially with respect to the tails of the distributions. For an a priori determination of the amount of fuel required, the most useful piece of data is the distribution function which can be used to determine how much velocity correction must be provided for a desired probability of success. (A 99-percent probability of success here means that there will be enough fuel to satisfy velocity correction demands in all but 1 percent of flights having the same characteristics.) The designer is here always concerned with the tails of the distributions. Thus, a rather large Monte Carlo sample is required if the sample data from which the tail characteristics are determined are to be statistically significant.

One of the principal results to be obtained from the Monte Carlo statistics is an indication of how conservative is the use of the gaussian uncorrelated assumption frequently employed to determine the fuel requirements. Specifically, we would like to determine: (1) How much greater than the true rms ΔV is the sum of the rms values of the individual corrections, and (2) how much less than three times the rms error figure is required to assure 99.74 percent mission success? The data for answering these questions is compiled in table II for each of the four situations simulated. The correction velocities corresponding to 68.26-, 95.44-, and 99.74-percent probabilities are taken from the appropriate ΔV distribution curves. Also given are the Monte Carlo rms figures, ΔV_{rms}; the sums of the individual correction rms values, Σv_{rms}; and three times the latter.

It is seen that the difference between Σv_{rms} and ΔV_{rms} ranges from 3.3 percent to 12.1 percent for the four cases, the largest difference, as expected, being in the situation where correlation between successive corrections is most pronounced - that is, the large correction error case. The smallest percentage difference is in the case with five-times-nominal injection errors, the reason being that here the ΔV is dominated by the first correction and correlation is therefore relatively less significant.

The difference between the 99.74 percent and three Σv_{rms} figures is seen to be substantial, ranging as high as 43.3 percent. This demonstrates the error which would be made in assuming $\Delta V_{rms} = \Sigma v_{rms}$ and assuming a gaussian distribution. The conclusion is that if fuel tankage is designed by such a rule, an unnecessary reserve will be allowed. Or, looking at it another way, the probability of mission success will be substantially greater than specified.

Comparing the data in table II in another way, we see that the 99.74-percent points range from 2.1 to 2.7 times the Σv_{rms} figures. Thus, a simple rule for using Σv_{rms} data to determine fuel requirements might be to multiply Σv_{rms} by, say, 2.4 to obtain approximately the 99.74-percent point on the ΔV probability distribution curve. It was pointed out previously that it has been the practice to multiply by 3 to obtain this figure, ostensibly on the basis that the probability density curve had a gaussian distribution. Since the curve is not gaussian the multiplying factor is less than 3. With a factor of 2.4, a saving of 20 percent in the fuel required for this portion of the mission may be effected. Depending on the specific situation regarding the magnitudes of injection and velocity correction errors, the savings may be substantially more or less than 20 percent, which is to say simply that a well-defined error model for the system must be given before system performance can be finally specified.

To complete the discussion of the Monte Carlo results, some observations should be made regarding the statistical significance of the data. A test which is fairly simple to apply determines the significance of the observed differences between the sample and theoretical rms values shown in table I for individual velocity corrections. This test involves computing the expected variations, or standard deviations, of the sample rms values and comparing these with the observed variations. First we define the squared magnitude of a particular velocity correction as a new random variable, $Y = |v|^2$. The sample mean-square value obtained from a Monte Carlo run is another random variable.

$$\overline{Y} = \frac{\Sigma |v|^2}{N}$$

where N is the number of Monte Carlo samples of v. Now, by the central limit theorem, we can assume that $\overline{Y}$ is normally distributed with mean and standard deviation given by

$$M_{\overline{Y}} = E(Y)$$

$$\sigma_{\overline{Y}}^2 = \frac{E(Y^2) - [E(Y)]^2}{N}$$

We, of course, know the value of N used in a particular Monte Carlo run. We also know the covariance matrix of v, from which, because of the assumed (multivariate) normal distribution of v, we can compute the expected values $E(Y)$ and $E(Y^2)$.

Performing these calculations for a specific velocity correction, namely v_1 for the nominal situation, we find that $M_{\overline{Y}} = 69.71$ and $\sigma_{\overline{Y}} = 1.378$. The sample value $\overline{Y}$ for this case is 70.52, which differs from its expected (or mean) value by 0.81, or about 0.6 of one standard deviation. Thus, the agreement between theory and the observation $\overline{Y}$ is satisfactory. Since the $\sigma_{\overline{Y}}$ is about 2 percent of $M_{\overline{Y}}$, it follows that the sample rms value, $\sqrt{\overline{Y}}$, will have a standard deviation on the order of half of this, or 1 percent. This certainly is a small enough uncertainty for this statistic so that the rms figures given in table I are reasonably reliable measures of the theoretical values.

CONCLUDING REMARKS

The results shown indicate that the Monte Carlo method is a practical technique for obtaining the statistics of midcourse velocity corrections. Obviously, extensions are possible to various other (approximately linear) problems in which covariance matrices have been computed.

It should be noted that the problem formulation employed in this paper includes the assumption of linear guidance law matrices, B_i. If the technique described here were to be used for systems in which the guidance calculations are nonlinear, it would be necessary to determine a linear approximation to the guidance law to obtain the appropriate B_i matrices. Such a procedure is seen to be relatively straightforward if one recognizes that in the linearized problem the elements of B_i are simply partial derivatives of the velocity correction components with respect to the vehicle position and velocity state variables. In some cases, analytical expressions for these partials can be developed; in other cases, perturbation techniques can be employed in the computer simulation of the problem to obtain the partials numerically.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., March 2, 1964

APPENDIX

THE VELOCITY CORRECTIONS EXPRESSED AS LINEAR FUNCTIONS OF A SET OF UNCORRELATED RANDOM VARIABLES

In the statement of the midcourse guidance problem, the ith velocity correction is written as a linear function of the estimated state, $\hat{x}_i$, and the correction mechanization error, v_{ce_i}:

$$v_i = B_i\hat{x}_i + v_{ce_i} \tag{A1}$$

The v_{ce_i} and $\hat{x}_i$ are assumed uncorrelated with each other in the error model herein employed. However, in general, these quantities will not be uncorrelated with corresponding components of the other velocity corrections. For the sake of facilitating the design of a Monte Carlo statistical analysis procedure, we would like to "orthogonalize" the set of random variables $\hat{x}_1$, v_{ce_1}, ... in the sense of finding an equivalent set of uncorrelated random variables in terms of which the v_i could be expressed.

In linear algebra the Gram-Schmidt procedure for orthogonalizing a set of linearly independent vectors belonging to an inner product space is well known. The same method can be used to orthogonalize a set of random variables by employing the covariance matrix of two random vectors x and y in the role of the inner product:

$$(x,y) = E[xy^T] \tag{A2}$$

where x is said to be "orthogonal" to y if $(x,y) = 0$, that is, if the covariance matrix $E[xy^T]$ is zero, or x and y are "uncorrelated." Although we do not have here quite the situation in which one usually employs the Gram-Schmidt procedure, we can still use essentially this technique.

First we need to establish some properties of the above function, (x,y). From known properties of the expectation and matrix operators, it is seen that

$$\left.\begin{array}{ll} \text{i)} & (x,y) = (y,x)^T \\ \text{ii)} & (x + y,z) = (x,z) + (y,z) \\ \multicolumn{2}{l}{\text{If } A \text{ is a matrix of constants, then}} \\ \text{iii)} & (Ax,y) = A(x,y) \\ & (x,Ay) = (x,y)A^T \end{array}\right\} \tag{A3}$$

Now let $w_1, w_2, \ldots, w_n$ be random vectors, and construct another set of random vectors, $y_1, \ldots, y_n$ in the following way:

$$\left.\begin{aligned}
y_1 &= w_1 \\
y_2 &= w_2 - (w_2, y_1)(y_1, y_1)^{-1} y_1 \\
y_3 &= w_3 - (w_3, y_2)(y_2, y_2)^{-1} y_2 - (w_3, y_1)(y_1, y_1)^{-1} y_1 \\
&\cdot \\
&\cdot \\
&\cdot \\
y_n &= w_n - (w_n, y_{n-1})(y_{n-1}, y_{n-1})^{-1} y_{n-1} \\
&\quad - \cdot \cdot \cdot \cdot (w_n, y_1)(y_1, y_1)^{-1} y_1
\end{aligned}\right\} \tag{A4}$$

Although (x,y) is not an inner product, it is seen from (A3) that it possesses all the necessary properties of an inner product which we require here, and the procedure represented by (A4) is conceptually nothing but the Gram-Schmidt process.

Note that it is necessary in (A4) for the $(y_i, y_i)^{-1}$ to exist for $i = 1, \ldots, n - 1$. Since $(y_i, y_i) = E[y_i y_i^T]$ is the covariance matrix of y_i, which is positive definite (except for certain exceptional instances which require special treatment), $(y_i, y_i)^{-1}$ will always exist.

If the process (A4) is conceptually the same as the Gram-Schmidt procedure, then we should expect that the random vectors $y_1, \ldots, y_n$ are orthogonal (i.e., uncorrelated). We can prove that this is so by showing that for every m,

$$(y_m, y_l) = 0 , \qquad l = 1, 2, \ldots m - 1 \tag{A5}$$

Beginning with $m = 2$, we have

$$\left.\begin{aligned}
(y_2, y_1) &= E[w_2 - (w_2, y_1)(y_1, y_1)^{-1} y_1] \, [y_1^T] \\
&= E w_2 y_1^T - (w_2, y_1)(y_1, y_1)^{-1} \, E[y_1 y_1^T] \\
&= E w_2 y_1^T - E w_2 y_1^Y = 0
\end{aligned}\right\} \tag{A6}$$

For $m = 3$, we find by applying the same method and utilizing (A6) that both (y_3, y_1) and (y_3, y_2) are zero. Continuing in the same manner, it is readily shown that for every m, $(y_m, y_l) = 0$ for all $l < m$, which is what we set out to prove. By interchanging the indices l and m it is seen that this result implies that

$$(y_l, y_m) = 0 \quad \text{for all} \quad l \neq m \tag{A7}$$

Thus, the random vectors $y_1, \ldots, y_n$ are orthogonal (uncorrelated) as expected.

In order to apply the formulas (A4) to our specific problem, we must be able to write expressions for covariance matrices, such as $(\hat{x}_i, v_{ce_j})$, $(\hat{x}_i, \hat{x}_j)$, etc., which express the correlation between the pairs of random variables. To find expressions for these matrices, we must go back to the theory of linear optimal estimation as given in references 1 and 3. The development proceeds as follows.

We begin by showing that $\Delta\hat{x}_i$, the change in the estimate due to a set of observations between the (i - 1) and ith velocity corrections, is uncorrelated with the estimate at the beginning of the interval. Assume a set of k observations. The estimation equation for the jth observation is

$$\hat{x}_j^*(t_j) = \hat{x}_{j-1}^*(t_j) + K_j[y_j - M_j\hat{x}_{j-1}^*(t_j)] \tag{A8}$$

or

$$\left.\begin{aligned} \Delta\hat{x}_j^* &= \hat{x}_j^* - \hat{x}_{j-1}^* \\ &= K_j[y_j - M_j\hat{x}_{j-1}^*] \end{aligned}\right\} \tag{A9}$$

where $\hat{x}_j^*$ is the estimate (based on j observations) of the state vector augmented to include the correlated observation errors (see refs. 1 and 3). The y_j is the observation,

$$\begin{aligned} y_j &= M_j x^*(t_j) \\ &= H_j x(t_j) + n(t_j) \end{aligned} \tag{A10}$$

where x is the vector of vehicle positions and velocities, and n is the additive observation error.

Now using equation (A10) in equation (A9), and invoking the definition of estimation error, $\tilde{x}^* = x^* - \hat{x}^*$, we obtain

$$\Delta\hat{x}_j^* = K_j M_j \tilde{x}_{j-1}^*(t_j) \tag{A11}$$

Now the propagation of estimation error between observations is given by

$$\tilde{x}_{j-1}^*(t_j) = \Phi^*(t_j, t_{j-1})\tilde{x}_{j-1}^*(t_{j-1}) + n_j' \tag{A12}$$

where

$$n_j' = \int_{t_{j-1}}^{t_j} \Phi_n(t_j, \tau)u_n(\tau)d\tau \tag{A13}$$

The Φ^* is the transition matrix for the augmented system in which the observation errors are regarded as additional state variables; Φ_n is the appropriate submatrix of Φ^*, and is dimensioned so that the n_j' adds only to that part of $\tilde{x}^*$ associated with the observation errors.

Substitution of (A12) into (A11) gives an equation for $\Delta\hat{x}_j^*$ in terms of $\tilde{x}_{j-1}^*$ and n_j':

$$\Delta\hat{x}_j^* = K_j M_j \Phi^*(t_j, t_{j-1})\tilde{x}_{j-1}^*(t_{j-1}) + K_j M_j n_j' \tag{A14}$$

By induction, it is apparent that $\Delta\hat{x}_j^*$ can be expressed as a linear function of the $\tilde{x}_0^*$ vector at the beginning of the sequence of observations and the j vectors $n_1', \ldots, n_j'$. Likewise, the total change in the estimate due to observations in the interval is

$$\Delta\hat{x}_i^* = \sum_{j=1}^{k} \Phi^*(t_i, t_j)\Delta\hat{x}_j^* \tag{A15}$$

and this is therefore a linear function of $\tilde{x}_0^*$ and $n_1', \ldots, n_k'$. Now, the property of an optimal linear estimate is that the estimate and error in estimate are orthogonal (refs. 1 and 3); that is,

$$E\hat{x}_0^* \tilde{x}_0^{*T} = 0 \tag{A16}$$

Also, the quantities n_j' are uncorrelated with any random vector outside the interval (t_{j-1}, t_j), or

$$E\hat{x}_0^* n_j' = 0 , \qquad j = 1, \ . \ . \ ., \ k \tag{A17}$$

Hence, $\Delta\hat{x}_i^*$ is uncorrelated with $\hat{x}_0^*$, or

$$E[\hat{x}_{i-1}\Delta\hat{x}_i^T] = (\hat{x}_{i-1}, \Delta\hat{x}_i) = 0 \tag{A18}$$

By similar reasoning, we arrive at the conclusion that all the $\Delta\hat{x}_i$ are uncorrelated with each other, with any previous estimate, or with any previous correction mechanization error:

$$\left.\begin{aligned} (\Delta\hat{x}_i, \Delta\hat{x}_q) &= 0 , \qquad i \neq q \\ (\Delta\hat{x}_i, \hat{x}_q) &= 0 , \qquad q < i \\ (\Delta\hat{x}_i, v_{ce_q}) &= 0 , \qquad q < i \end{aligned}\right\} \tag{A19}$$

Also, by assumption, $(\hat{x}_i, v_{ce_i}) = 0$.

Returning to the application of the expressions (A4) to the midcourse velocity correction problem, we can now write:

$$y_1 = \hat{x}_1 = \Delta\hat{x}_1 \tag{A20}$$

$$y_2 = v_{ce_1} - (\hat{x}_1, v_{ce_1})(v_{ce_1}, v_{ce_1})^{-1} v_{ce_1} = v_{ce_1} \tag{A21}$$

$$y_3 = \hat{x}_2 - (\hat{x}_2, v_{ce_1})(v_{ce_1}, v_{ce_1})^{-1} v_{ce_1} - (\hat{x}_2, \Delta\hat{x}_1)(\Delta\hat{x}_1, \Delta\hat{x}_1)^{-1} \Delta\hat{x}_1 \tag{A22}$$

The covariance matrices $(\hat{x}_2, v_{ce_1})$ and $(\hat{x}_2, \Delta\hat{x}_1)$ are evaluated by applying the relation (from eq. (15) in the text):

$$\left.\begin{aligned} \hat{x}_2 &= \Phi_2 \hat{x}_1' + \Delta\hat{x}_2 \\ &= \Phi_2(I + B_1)\Delta\hat{x}_1 + \Phi_2 v_{ce_1} + \Delta\hat{x}_2 \end{aligned}\right\} \tag{A23}$$

Since $(\Delta\hat{x}_2, v_{ce_1})$, $(v_{ce_1}, \Delta\hat{x}_1)$, and $(\Delta\hat{x}_2, \Delta\hat{x}_1)$ are zero from the preceding development, we find

$$\left.\begin{aligned} (\hat{x}_2, v_{ce_1}) &= \Phi_2(v_{ce_1}, v_{ce_1}) \\ (\hat{x}_2, \Delta\hat{x}_1) &= \Phi_2(I + B_1)(\Delta\hat{x}_1, \Delta\hat{x}_1) \end{aligned}\right\} \tag{A24}$$

Hence,

$$\left.\begin{aligned} y_3 &= \hat{x}_2 - \Phi_2 v_{ce_1} - \Phi_2(I + B_1)\Delta\hat{x}_1 \\ &= \Delta x_2 \end{aligned}\right\} \tag{A25}$$

For y_4, we then have

$$\begin{aligned} y_4 = v_{ce_2} &- (v_{ce_2}, \Delta\hat{x}_2)(\Delta\hat{x}_2, \Delta\hat{x}_2)^{-1}\Delta\hat{x}_2 \\ &- (v_{ce_2}, v_{ce_1})(v_{ce_1}, v_{ce_1})^{-1} v_{ce_1} \\ &- (v_{ce_2}, \Delta\hat{x}_1)(\Delta\hat{x}_1, \Delta\hat{x}_1)^{-1}\Delta\hat{x}_1 \end{aligned} \tag{A26}$$

where $(v_{ce_2}, \Delta\hat{x}_2)$, (v_{ce_2}, v_{ce_1}), and $(v_{ce_2}, \Delta\hat{x}_1)$ are zero, and

$$y_4 = v_{ce_2} \tag{A27}$$

Then for y_5, we have

$$\begin{aligned} y_5 = \hat{x}_3 &- (\hat{x}_3, v_{ce_2})(v_{ce_2}, v_{ce_2})^{-1} v_{ce_2} \\ &- (\hat{x}_3, \Delta\hat{x}_2)(\Delta\hat{x}_2, \Delta\hat{x}_2)^{-1}\Delta\hat{x}_2 \\ &- (\hat{x}_3, v_{ce_1})(v_{ce_1}, v_{ce_1})^{-1} v_{ce_1} \\ &- (\hat{x}_3, \Delta\hat{x}_1)(\Delta\hat{x}_1, \Delta\hat{x}_1)^{-1}\Delta\hat{x}_1 \end{aligned} \tag{A28}$$

To evaluate the covariance matrices, we substitute

$$\left.\begin{aligned}\hat{x}_3 &= \Phi_3(I + B_2)\hat{x}_2 + \Phi_3 v_{ce_2} + \Delta\hat{x}_3 \\ &= \Phi_3(I + B_2)\Phi_2(I + B_1)\Delta\hat{x}_1 + \Phi_3(I + B_2)\Phi_2 v_{ce_1} \\ &\quad + \Phi_3(I + B_2)\Delta\hat{x}_2 + \Phi_3 v_{ce_2} + \Delta\hat{x}_3\end{aligned}\right\} \tag{A29}$$

and we can then determine that

$$\left.\begin{aligned}(\hat{x}_3, v_{ce_2}) &= \Phi_3(v_{ce_2}, v_{ce_2}) \\ (\hat{x}_3, \Delta\hat{x}_2) &= \Phi_3(I + B_2)(\Delta\hat{x}_2, \Delta\hat{x}_2) \\ (\hat{x}_3, v_{ce_1}) &= \Phi_3(I + B_2)\Phi_2(v_{ce_1}, v_{ce_1}) \\ (\hat{x}_3, \Delta\hat{x}_1) &= \Phi_3(I + B_2)\Phi_2(I + B_1)(\Delta\hat{x}_1, \Delta\hat{x}_1)\end{aligned}\right\} \tag{A30}$$

From this it is evident that

$$y_5 = \Delta\hat{x}_3 \tag{A31}$$

Proceeding in like manner, we see that the required orthogonal random variables are $\Delta\hat{x}_1, v_{ce_1}, \ldots, \Delta\hat{x}_n, v_{ce_n}$, where n is the number of velocity corrections considered.

At this point it is easy to see the error which has been made in reference 4. In effect, the authors of reference 4 assume that the estimation error is uncorrelated with the actual state, and that the estimation error after $m + n$ observations is uncorrelated with that previously obtained from m observations. That this is not so is readily determined either by a direct application of certain relations developed above or by a unique argument to the effect that there is only one possible complete set of uncorrelated random variables, namely, the set given above.

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2. McLean, John D., Schmidt, Stanley F., and McGee, Leonard A.: Optimal Filtering and Linear Prediction Applied to a Midcourse Navigation System for the Circumlunar Mission. NASA TN D-1208, 1962.

3. Smith, Gerald L.: Secondary Errors and Off-Design Conditions in Optimal Estimation of Space Vehicle Trajectories. NASA TN D-2129, 1964.

4. Skidmore, Lionel J., and Penzo, Paul A.: Monte Carlo Simulation of the Midcourse Guidance for Lunar Flights. AIAA Journal, vol. 1, no. 4, April, 1963, pp. 820-31.

TABLE I.- SAMPLE STATISTICS

Condition statistic, meters/sec	Velocity correction					
	v_1	v_2	v_3	v_4	v_5	ΔV
No velocity correction error						
99-percent probability	21.45	3.14	1.94	5.22	0.36	27.65
Sample mean	6.74	.991	.695	1.69	.117	10.23
Sample rms	8.39	1.26	.841	2.08	.146	11.75
Theoretical rms	8.41	1.22	.813	2.05	.148	---
Nominal						
99-percent probability	21.50	3.82	2.75	6.31	3.41	29.33
Sample mean	6.74	1.22	.969	2.03	1.12	12.08
Sample rms	8.40	1.49	1.13	2.49	1.36	13.43
Theoretical rms	8.40	1.48	1.13	2.47	1.37	---
Two-times-nominal velocity correction error						
99-percent probability	21.41	3.47	3.64	7.29	7.16	31.73
Sample mean	6.75	1.18	1.33	2.43	2.32	14.01
Sample rms	8.41	1.40	1.54	2.94	2.82	15.26
Theoretical rms	8.41	1.39	1.53	2.91	2.83	---
Five times injection error						
99-percent probability	107.0	4.09	2.61	19.00	5.95	130.50
Sample mean	33.82	1.66	1.01	5.95	1.95	44.40
Sample rms	42.14	1.85	1.15	7.36	2.37	53.13
Theoretical rms	42.13	1.84	1.15	7.24	2.38	---

TABLE II.- EVALUATION OF THE "THREE-RMS" RULE

Statistic, meters/sec	Condition: No velocity correction error	Condition: Nominal	Condition: Two-times-nominal velocity correction error	Condition: Five-times-nominal velocity correction error
$\Delta V_{0.6826}$	12.02	13.95	16.17	53.7
$\Delta V_{0.9544}$	21.83	23.78	25.77	103.9
$\Delta V_{0.9974}$	31.25	33.40	35.80	148.0
ΔV_{rms}	11.75	13.43	15.26	53.13
Σv_{rms}	12.69	14.87	17.11	54.87
$3\Sigma v_{rms}$	38.07	44.61	51.33	164.61
$\Sigma v_{rms} - \Delta V_{rms}$	.94 (8.0%)	1.44 (10.7%)	1.85 (12.1%)	1.74 (3.3%)
$3\Sigma v_{rms} - \Delta V_{0.9974}$	6.82 (21.8%)	11.21 (33.6%)	15.53 (43.3%)	16.61 (11.2%)
$\dfrac{\Sigma \Delta V_{0.9974}}{v_{rms}}$	2.46	2.25	2.09	2.70

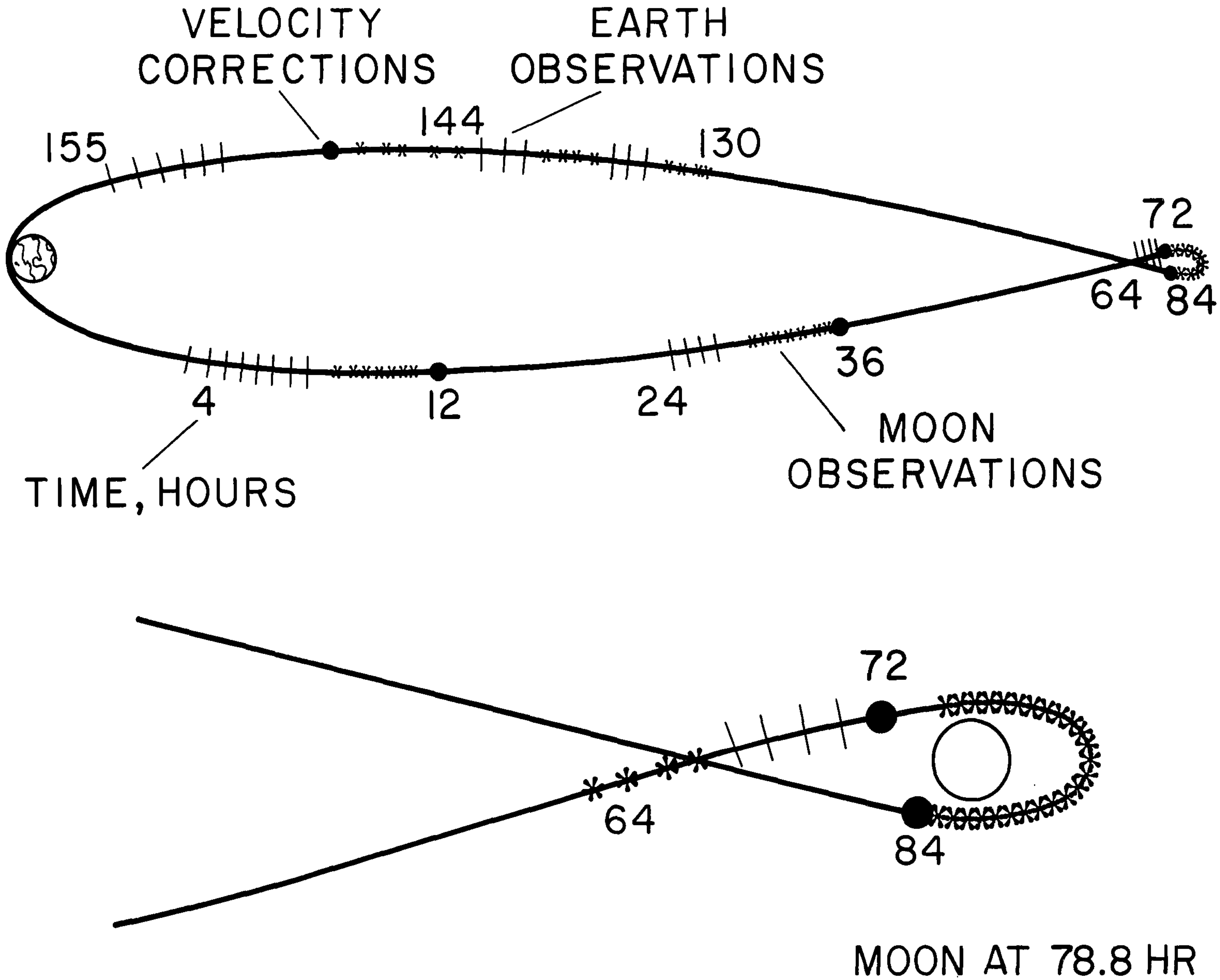
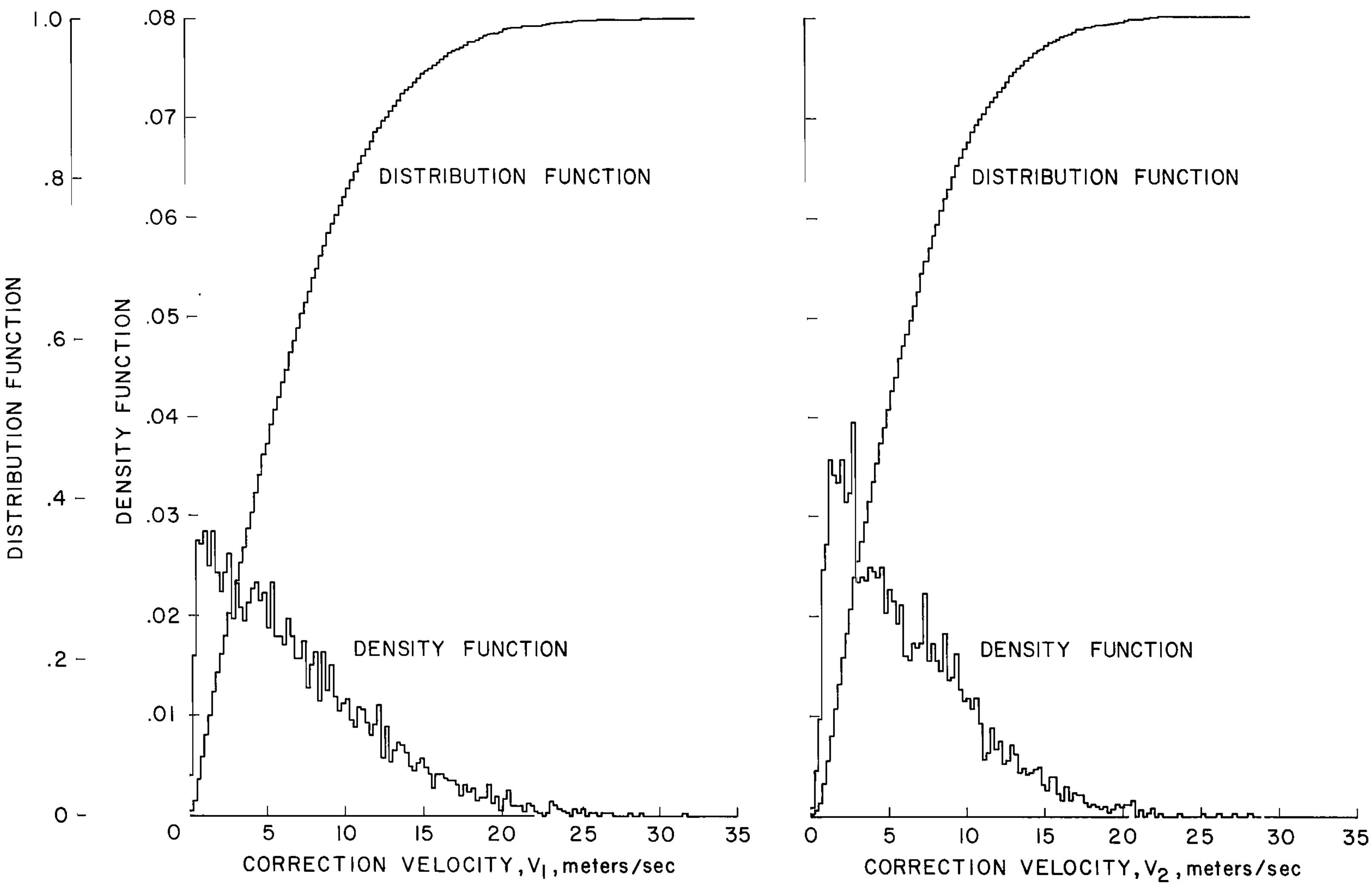


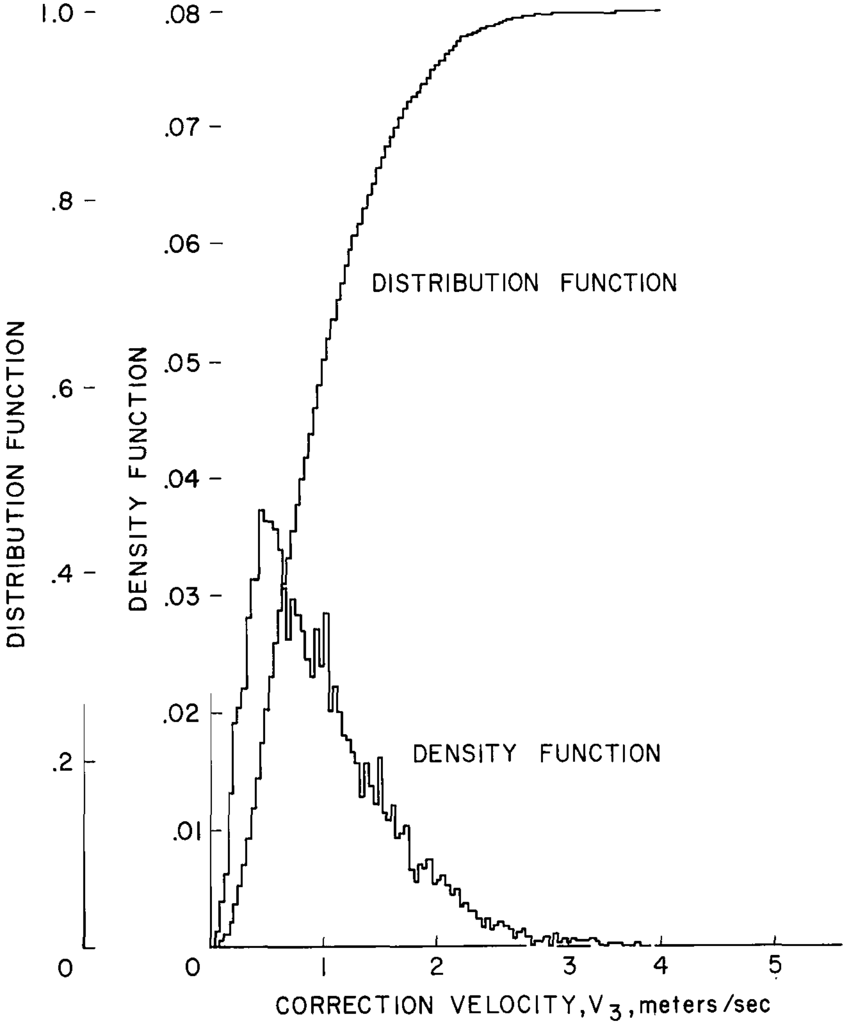
Figure 1.- Schedule of observations and velocity corrections.



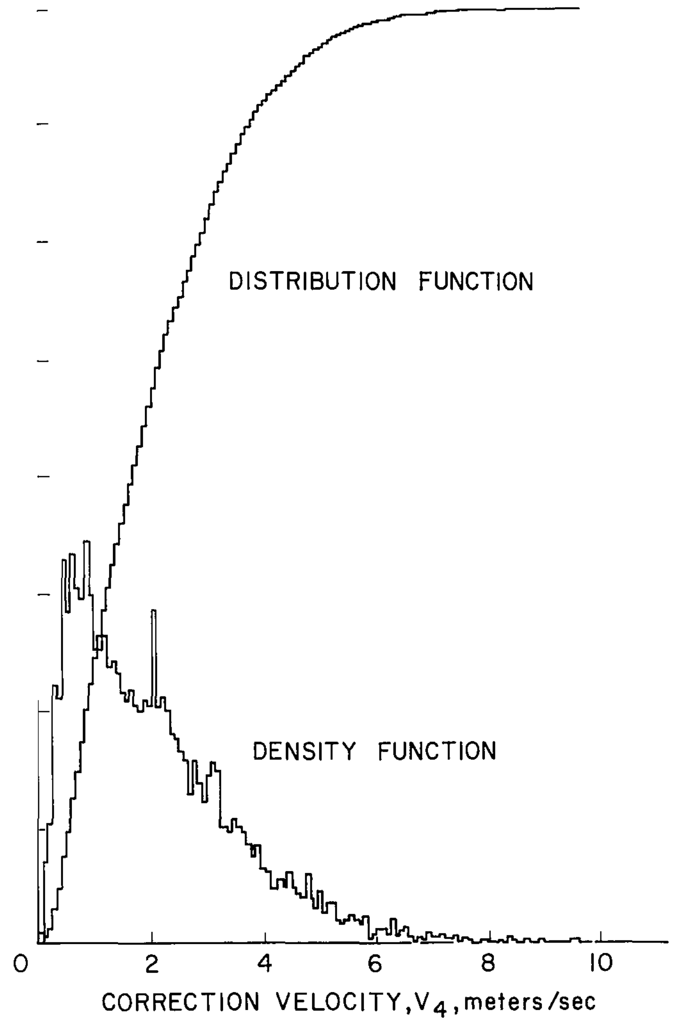
(a) First velocity correction.

(b) Second velocity correction.

Figure 2.- Statistical characteristics of the velocity corrections for the nominal case.

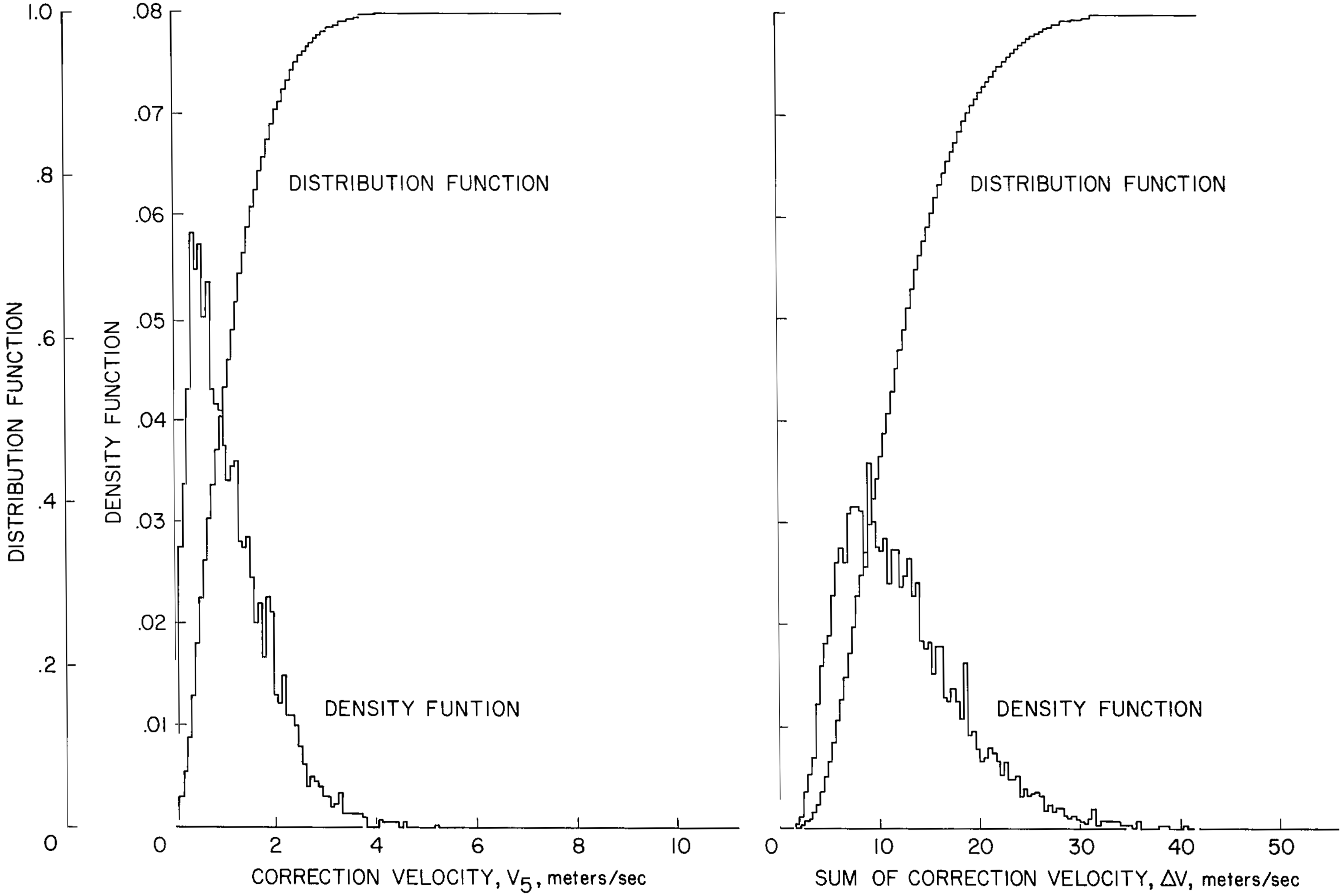


(c) Third velocity correction.



(d) Fourth velocity correction.

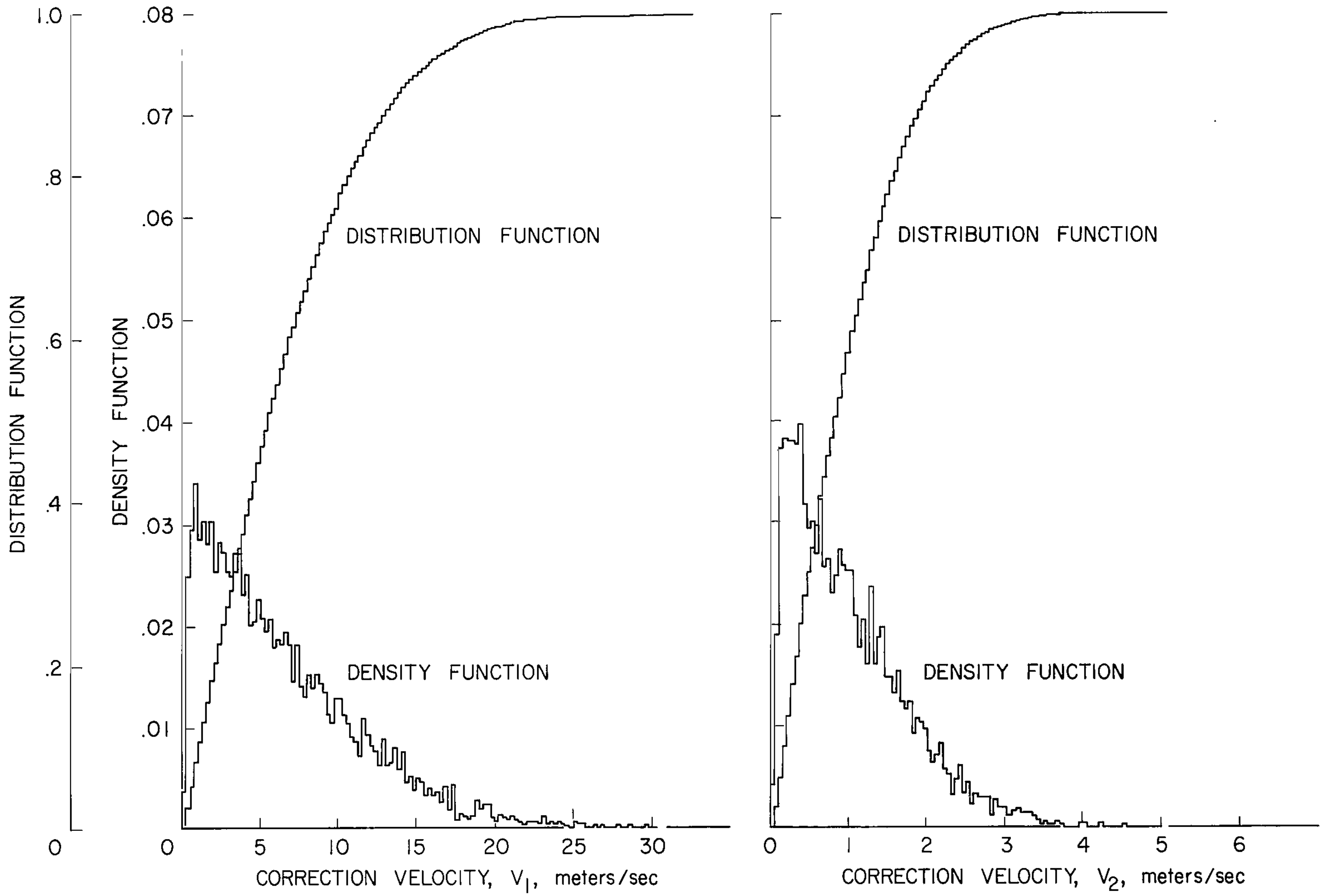
Figure 2.- Continued.



(e) Fifth velocity correction.

(f) Sum of velocity corrections.

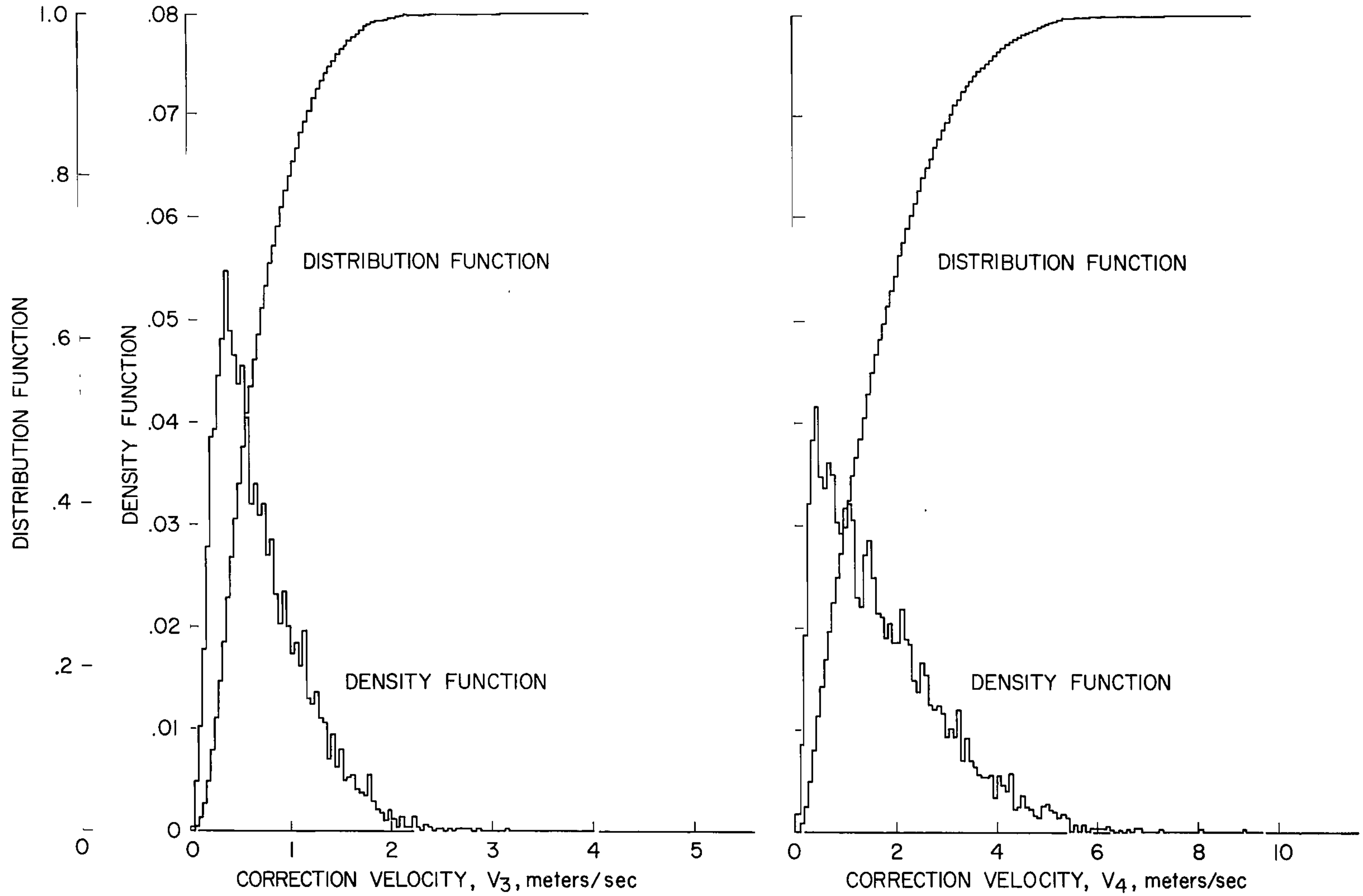
Figure 2.- Concluded.



(a) First velocity correction.

(b) Second velocity correction.

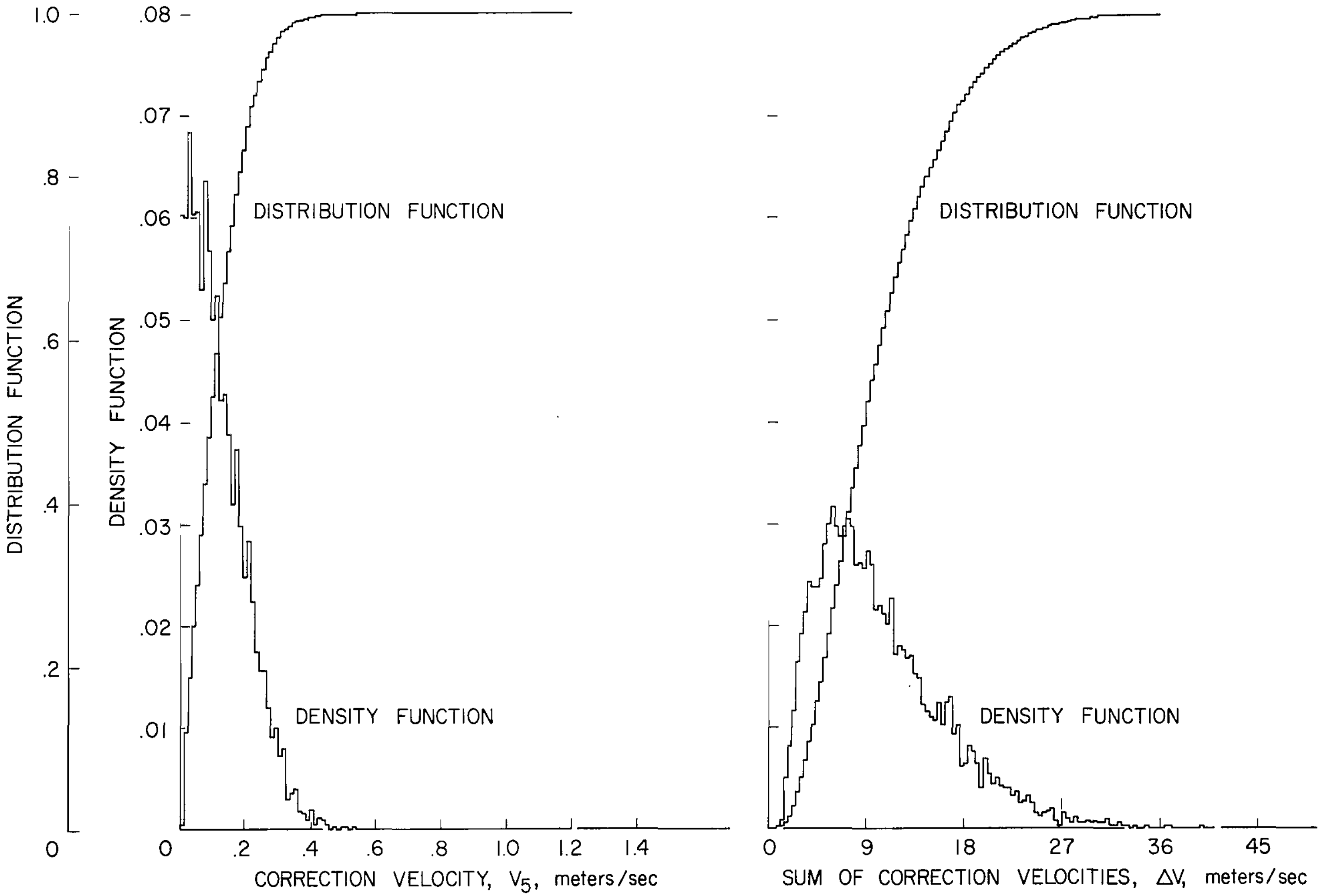
Figure 3.- Statistical characteristics of the velocity corrections for the case with no velocity correction errors.



(c) Third velocity correction.

(d) Fourth velocity correction.

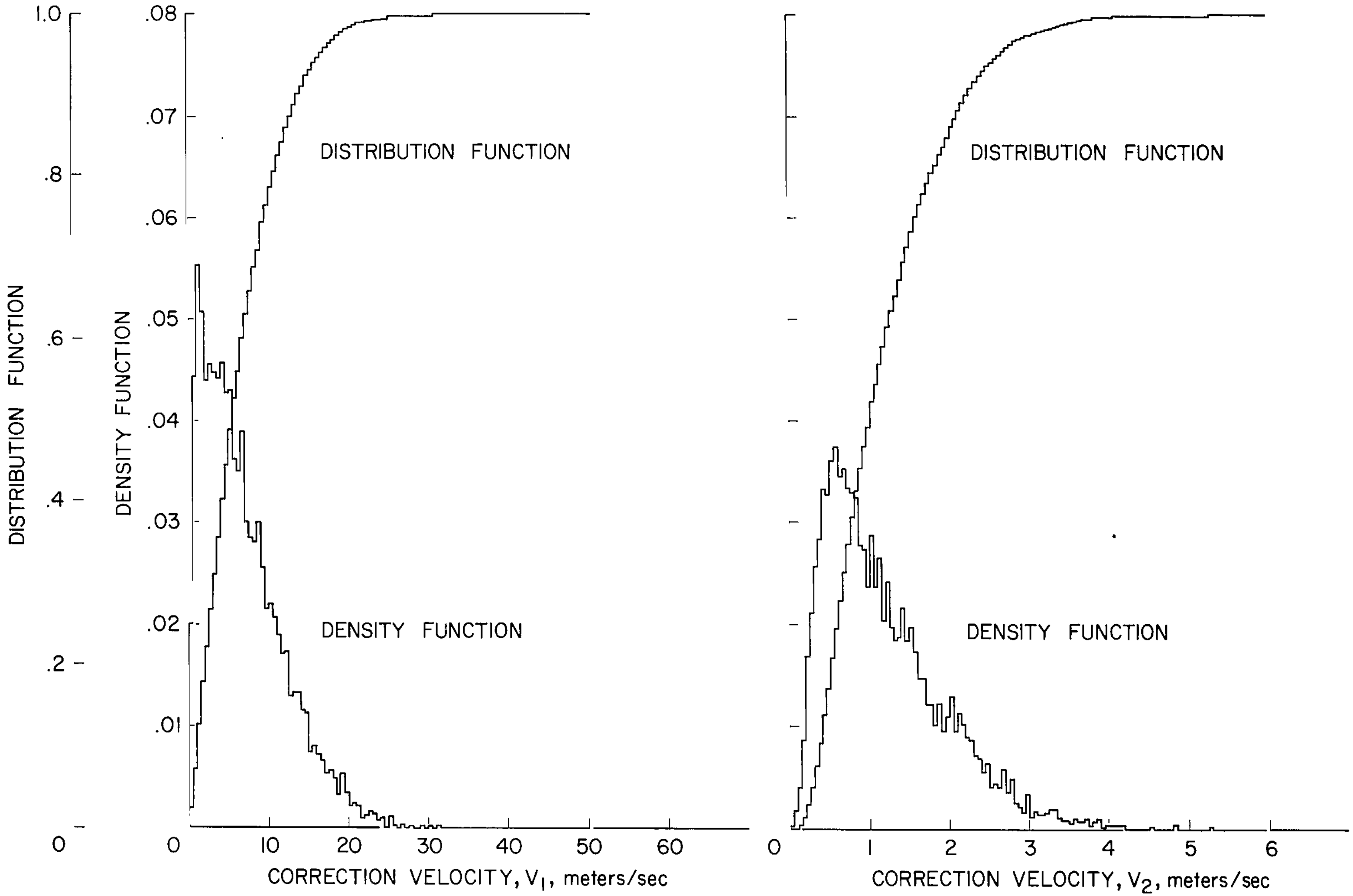
Figure 3.- Continued.



(e) Fifth velocity correction.

(f) Sum of velocity corrections.

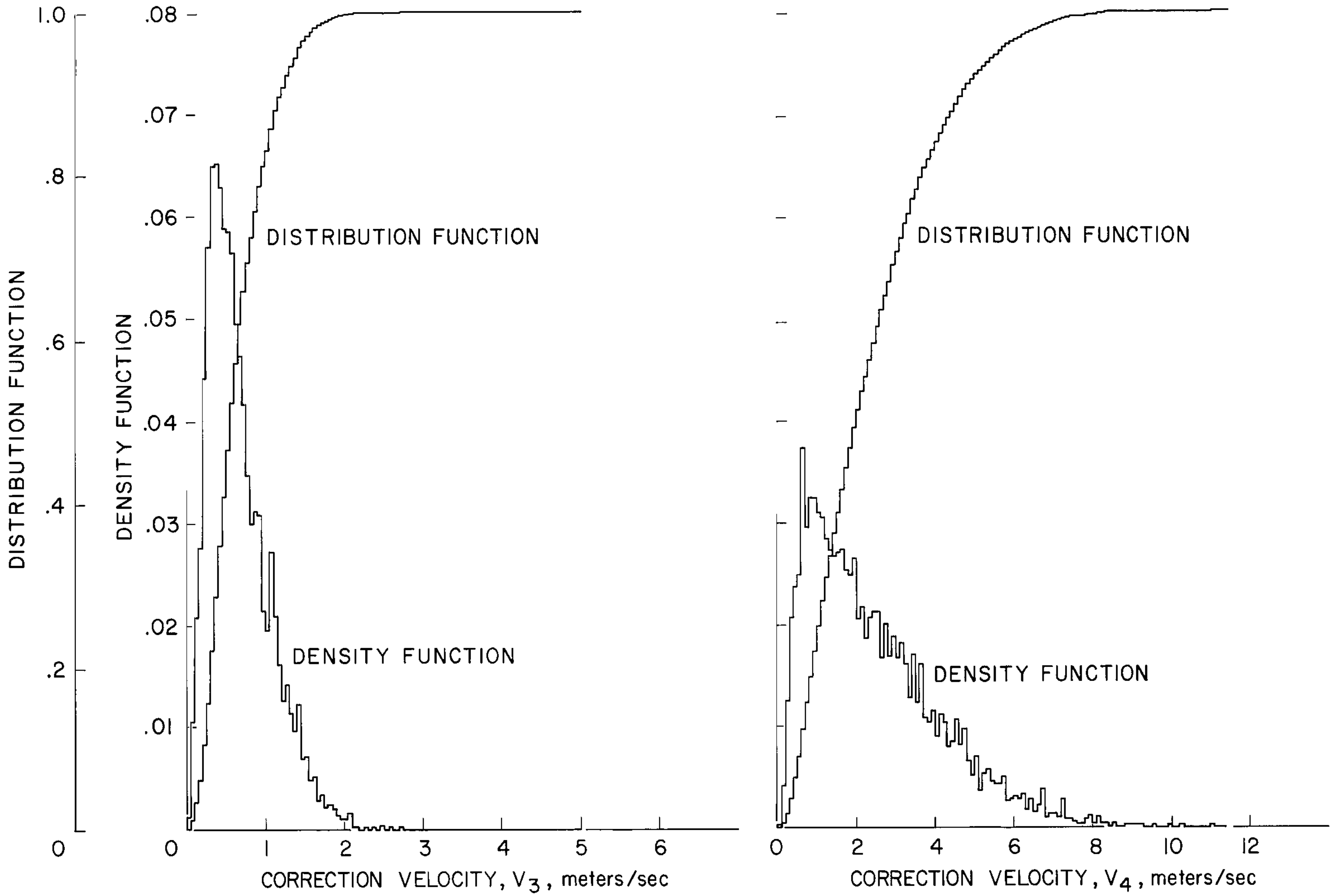
Figure 3.- Concluded.



(a) First velocity correction.

(b) Second velocity correction.

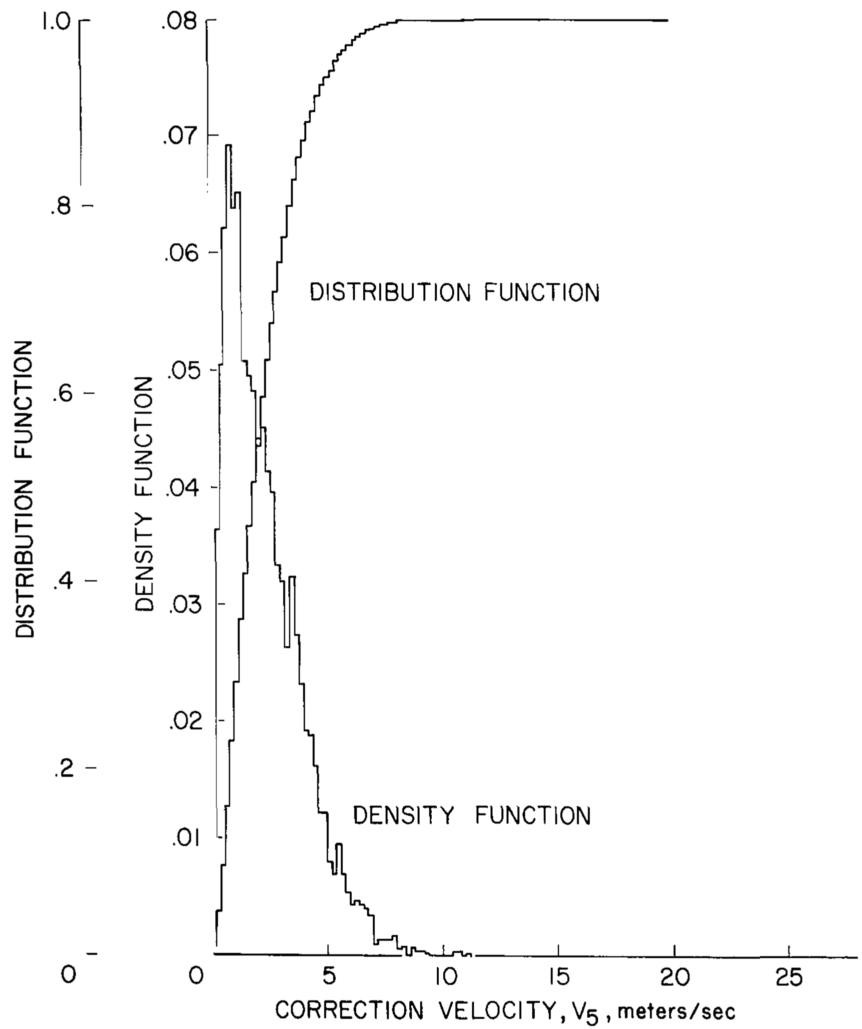
Figure 4.- Statistical characteristics of the velocity corrections for the case with twice nominal velocity correction errors.



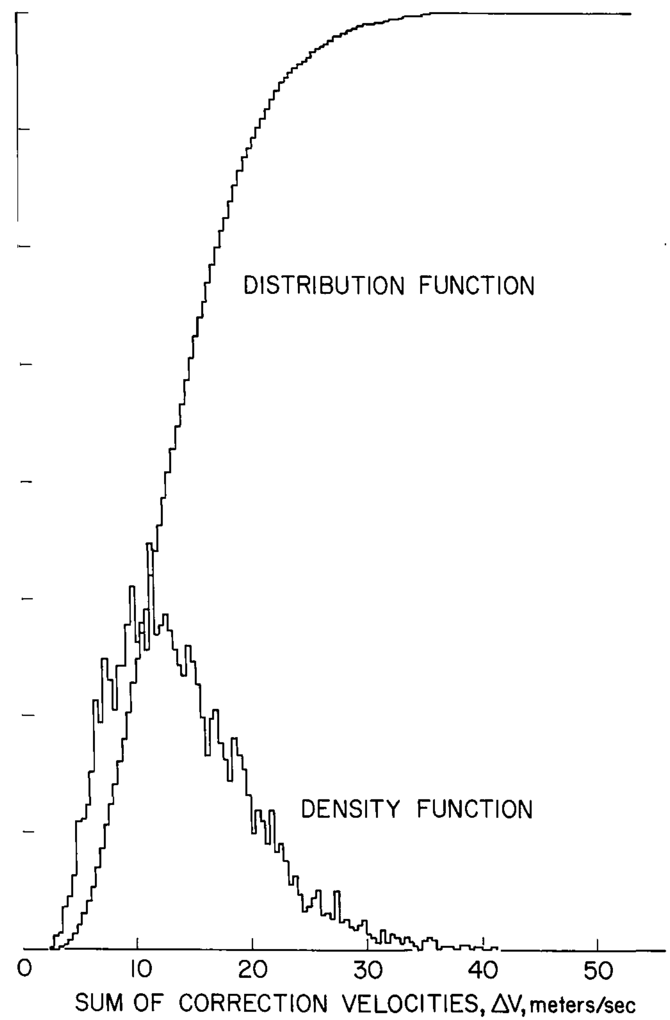
(c) Third velocity correction.

(d) Fourth velocity correction.

Figure 4.- Continued.

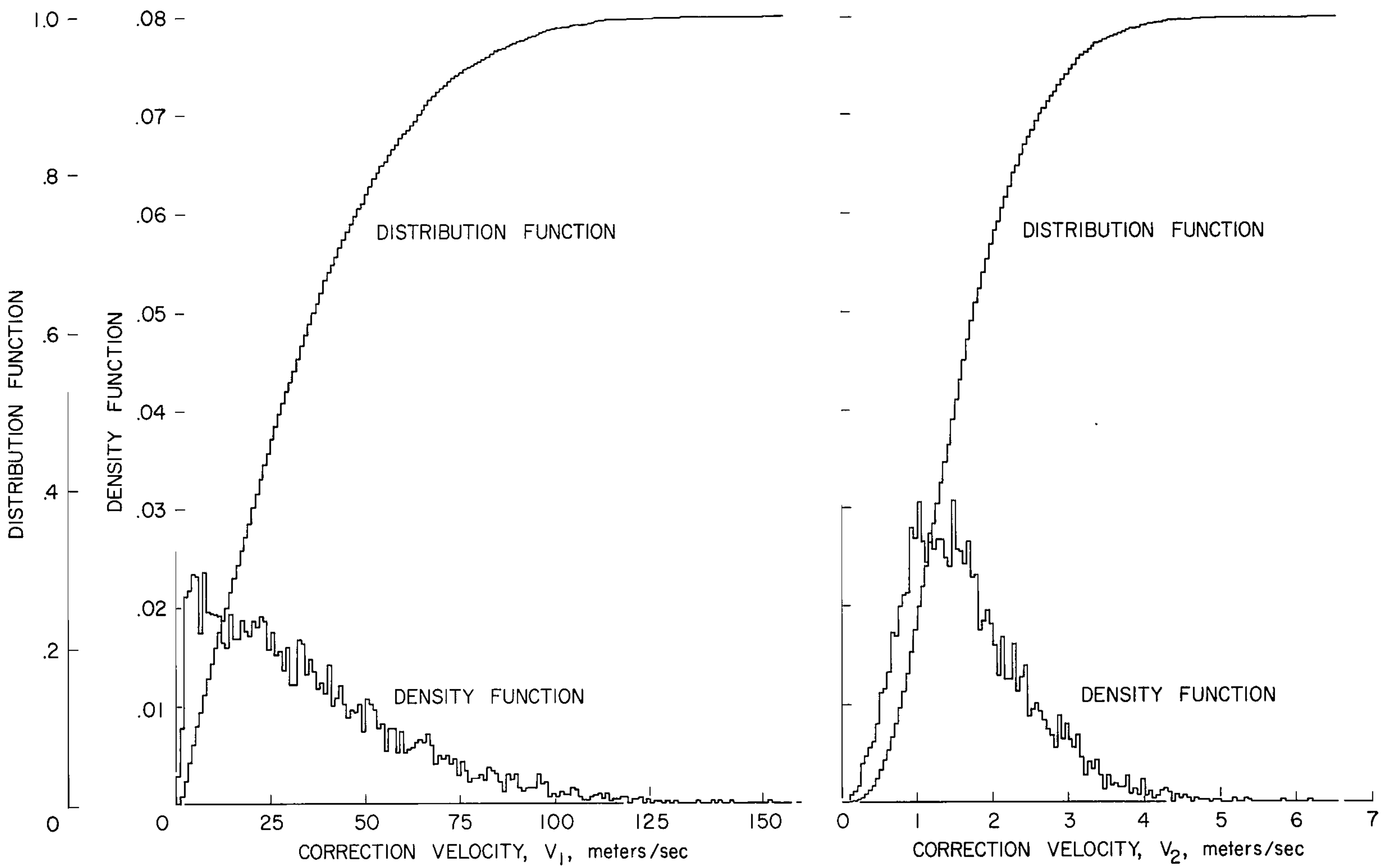


(e) Fifth velocity correction.



(f) Sum of velocity corrections.

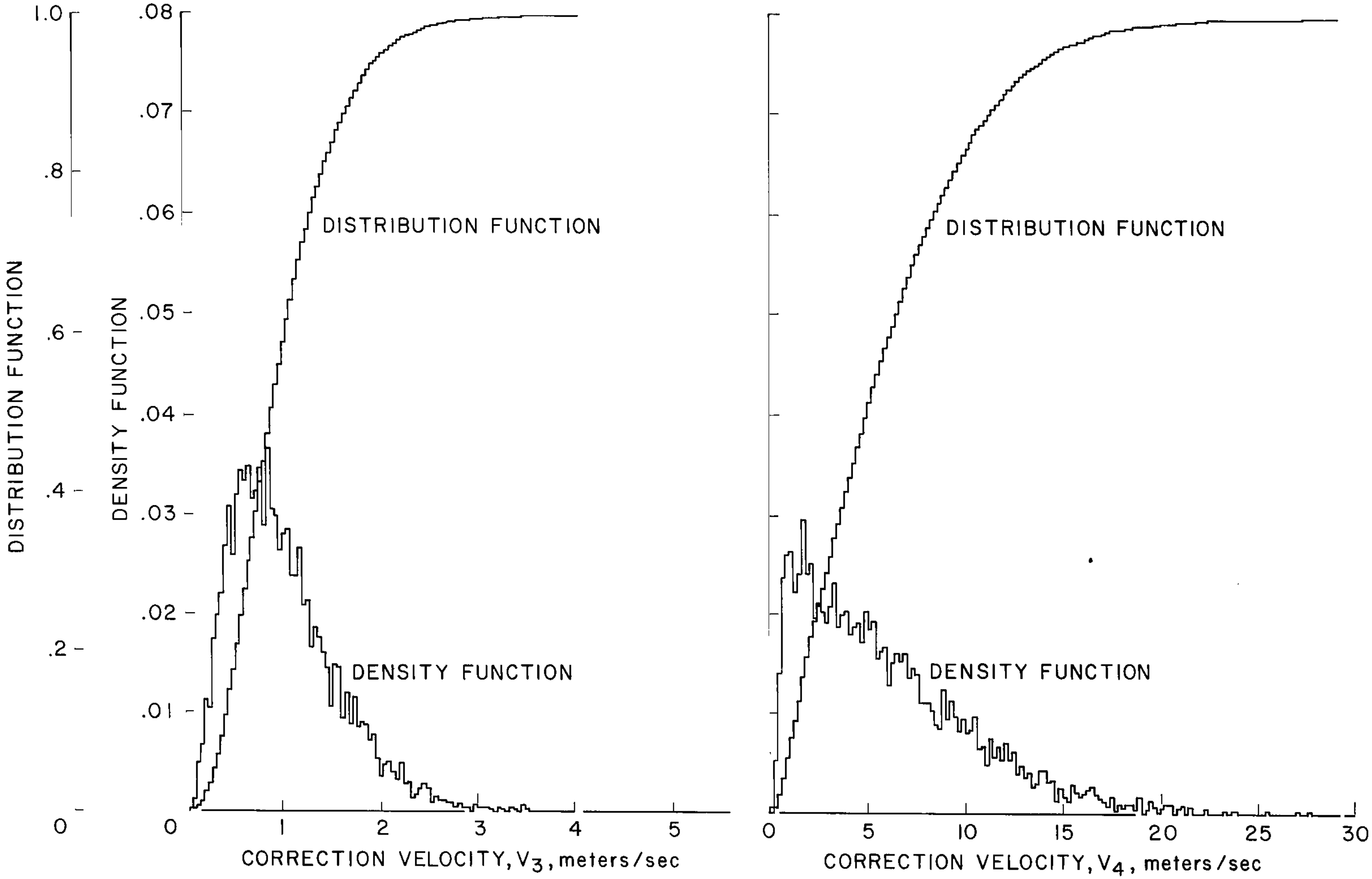
Figure 4.- Concluded.



(a) First velocity correction.

(b) Second velocity correction.

Figure 5.- Statistical characteristics of the velocity corrections for the case with five times nominal injection errors.



(c) Third velocity correction.

(d) Fourth velocity correction.

Figure 5.- Continued.

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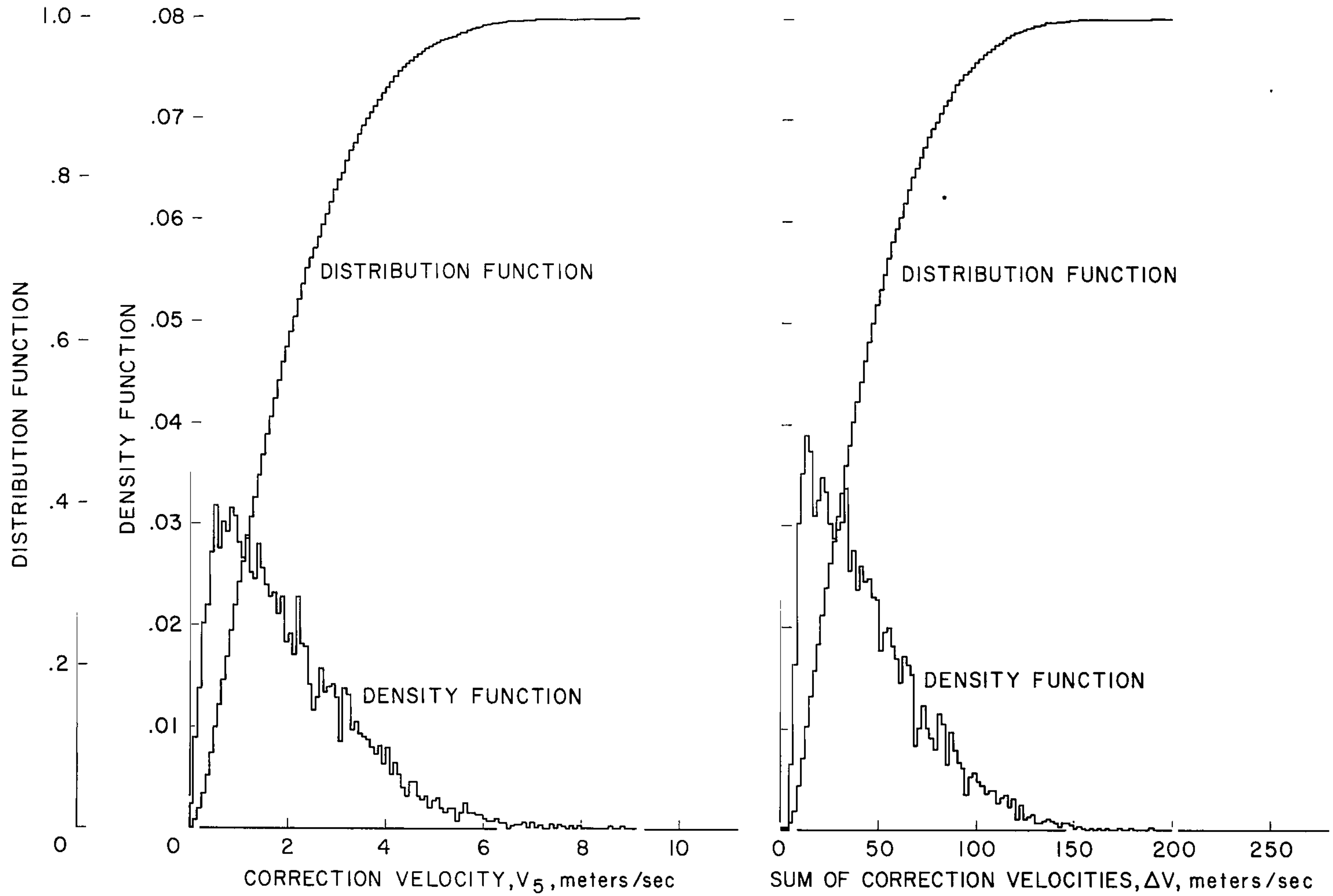

(e) Fifth velocity correction.

(f) Sum of velocity corrections.

Figure 5.- Concluded.

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